Variational and dissipative aspects of nonholonomic dynamics

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- **Nonholonomic systems and control**

- **Measure preservation and dissipation in nonholonomic systems**

- **Nonholonomic systems and fields**

In honor of Roger Brockett's 70th Birthday

•Background

• Basic observation about Hamiltonian systems: satisfy Liouville's theorem, preserving volume in phase space, thus cannot exhibit asymptotic stability.

Reflection of this: spectrum of linearization about a fixed point symmetric about imaginary axis.

Class of energy preserving systems which can exhibit asymptotic stability: nonholonomic systems – systems with nonintegrable constraints. In the absence of external dissipative forces, are always energy preserving.

Do not necessarily preserve volume in the phase space – – see for example Zenkov, Bloch and Marsden [1998], Zenkov and Bloch [2002], Kozlov, Jovanovich,

• Infinite Dimensions – oscillators interacting with fields. Hagerty, Bloch and Weinstein. Bloch, Hagerty, Rojo and Weinstein. Radiation Damping. Sofer and Weinstein. Original model of Lamb. Overall system Hamiltonian but can induce dissipation locally in oscillator.

Geometry and Kinematics of the Vertical Disk.

Configuration space: $Q = \mathbb{R}^2 \times S^1 \times S^1$, parameterized by coordinates $q = (x, y, \theta, \varphi)$.

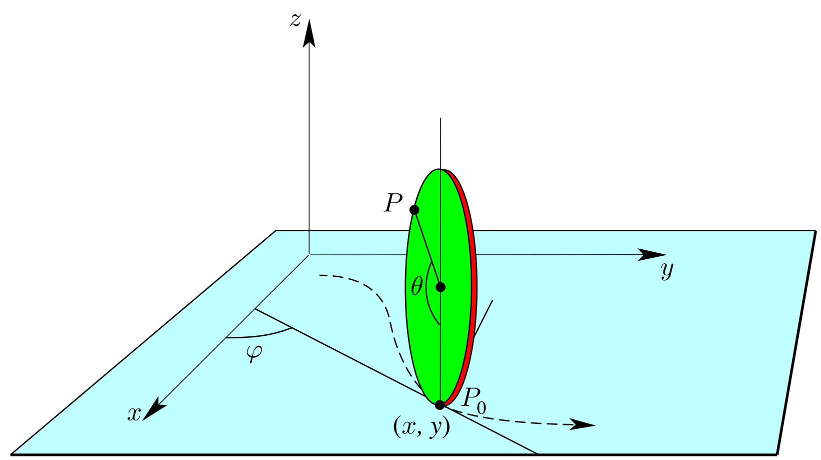

Figure 0.1: The geometry for the rolling disk.

The Lagrangian for the system: the kinetic energy

$$L(x, y, \theta, \phi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2.$$

If R is the radius of the disk, the nonholonomic constraints of rolling without slipping are

$$\begin{aligned} \dot{x} &= R(\cos\varphi)\dot{\theta} \\ \dot{y} &= R(\sin\varphi)\dot{\theta}, . \end{aligned}$$

Dynamics of the Controlled Disk. **Consider the case where we have two controls, one that can steer the disk and another that determines the roll torque.**

Lagrange d'Alembert equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = u_1 X_1 + u_2 X_2 + \lambda_1 W_1 + \lambda_2 W_2,$$

where

$$\frac{\partial L}{\partial \dot{q}} = (m\dot{x}, m\dot{y}, I\dot{\theta}, J\dot{\varphi})^T,$$

$$X_1 = (0, 0, 1, 0)^T, X_2 = (0, 0, 0, 1)^T,$$

and

$$W_1^T = (1, 0, -R\cos\varphi, 0), \quad W_2^T = (0, 1, -R\sin\varphi, 0)^T,$$

together with the constraint equations.

• Controlled case: controls u_1, u_2. Call the variables θ and ϕ "base" or "controlled" variables and the variables x and y "fiber" variables. θ and φ are controlled directly, the variables x and y are controlled indirectly via the constraints.

Clear base variables are controllable in any sense we can imagine. Is full system is controllable? Yes by virtue of the nonholonomic nature of the constraints.

The Kinematic Controlled Disk. In this case we imagine we have direct control over velocities rather than forces and, accordingly, we consider the most general first order system satisfying the constraints or lying in the "constraint distribution".

This system is

$$\dot{q} = u_1 \overline{X}_1 + u_2 \overline{X}_2$$

where $\overline{X}_1 = (\cos\varphi, \sin\varphi, 1, 0)^T$ **and** $\overline{X}_2 = (0, 0, 0, 1)^T$**.**

See Brockett nonholonomic integrator.

•Nonholonomic Equations of Motion

See e.g. Bloch, Krishnaprsad, Marsden and Murray [1996] and Zenkov, Bloch and Marsden [1998], Bloch and Crouch [1995] and other references in these papers.

•The Lagrange-d'Alembert Principle

• Consider a system with a configuration space Q, local coordinates q^i and m nonintegrable constraints

$$\dot{s}^a + A^a_\alpha(r, s)\dot{r}^\alpha = 0$$

where $q = (r, s) \in \mathbb{R}^{n-p} \times \mathbb{R}^p$, which we write as $q^i = (r^\alpha, s^a)$, where $1 \leq \alpha \leq n - p$ and $1 \leq a \leq p$.

• Lagrangian $L(q^i, \dot{q}^i)$.

Equations of motion given by Lagrange-d'Alembert principle.

Definition 0.1 ***The Lagrange-d'Alembert equations of motion for the system are those determined by***

$$\delta \int_a^b L(q^i, \dot{q}^i)\, dt = 0,$$

where we choose variations $\delta q(t)$ ***of the curve*** $q(t)$ ***that satisfy*** $\delta q(a) = \delta q(b) = 0$ ***and*** $\delta q(t)$ ***satisfies the constraints for each*** t ***where*** $a \leq t \leq b$***.***

- **This principle is supplemented by the condition that the curve itself satisfies the constraints.**

- **Note that we take the variation *before* imposing the constraints; that is, we do not impose the constraints on the family of curves defining the variation.**

• **The Falling Rolling Disk More realistic disk allowed to fall over.**

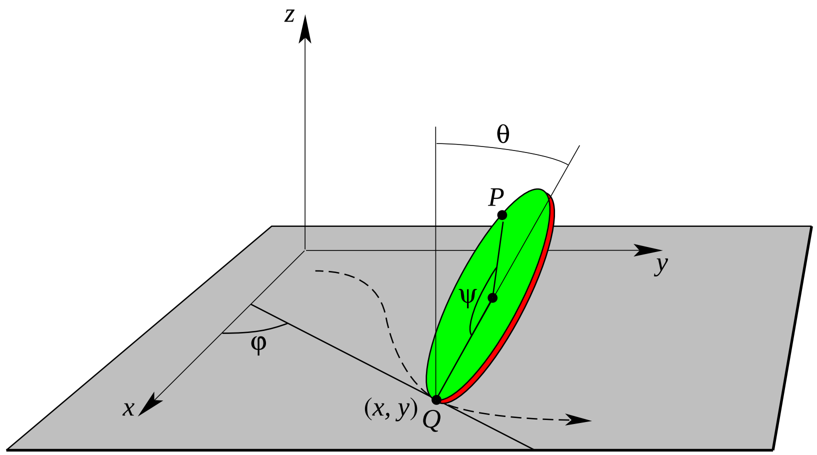

Figure 0.2: The geometry for the rolling disk.

This is a system which exhibits stability but not asymptotic stability.

•The Chaplygin Sleigh

Here we describe the Chaplygin sleigh, perhaps the simplest mechanical system which illustrates the possible dissipative nature of energy preserving nonholonomic systems.

Nonholonomic: subject to nonintegrable constraints – satisifes Lagrange D'Alembert equations.

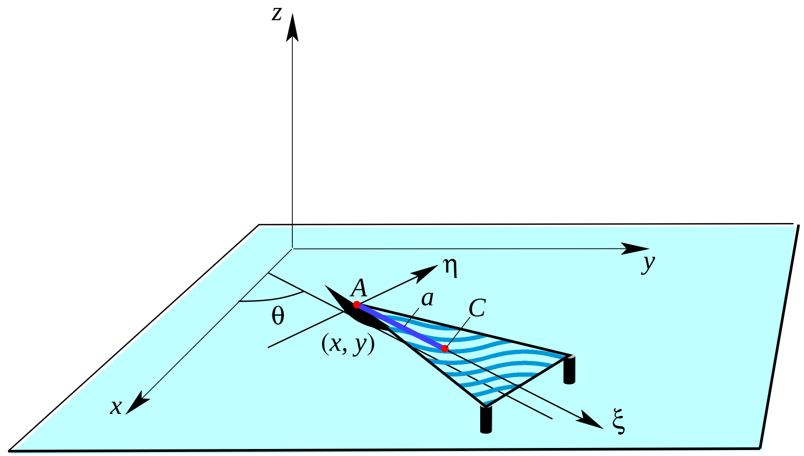

Figure 0.3: The Chaplygin sleigh is a rigid body moving on two sliding posts and one knife edge.

Equations:

$$\begin{aligned}\dot{v} &= a\omega^2 \\ \dot{\omega} &= -\frac{ma}{I + ma^2} v\omega\end{aligned}$$

Equations have a family of relative equilibria given by $(v, \omega) | v = \text{const}, \quad \omega = 0$**.**

Linearizing about any of these equilibria one finds one zero eigenvalue and one negative eigenvalue.

In fact the solution curves are ellipses in $v - \omega$ **plane with the positive** v**-axis attracting all solutions.**

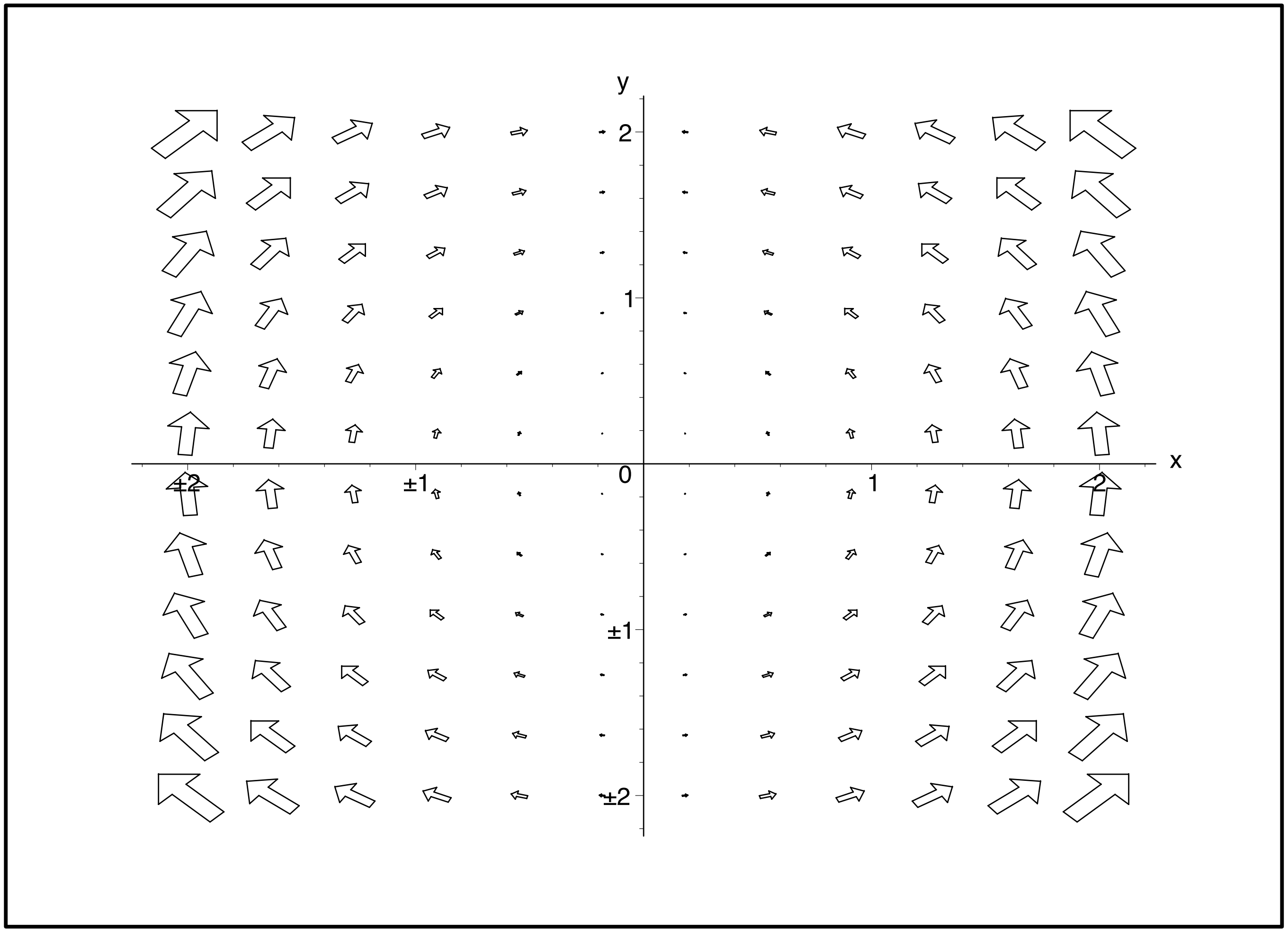

Figure 0.4: Chaplygin Sleigh/2d Toda phase portrait.

•Euler-Poincaré-Suslov Equations

Important special case of the reduced nonholonomic equations.

•Example: Euler-Poincaré-Suslov Problem on $SO(3)$ **In this case the problem can be formulated as the standard Euler equations**

$$I\dot{\omega} = I\omega \times \omega$$

where $\omega = (\omega_1, \omega_2, \omega_3)$ **are the system angular velocities in a frame where the inertia matrix is of the form** $I = \mathrm{diag}(I_1, I_2, I_3)$ **and the system is subject to the constraint**

$$a \cdot \omega = 0$$

where $a = (a_1, a_2, a_3)$**.**

The nonholonomic equations of motion are then given by

$$I\dot{\omega} = I\omega \times \omega + \lambda a$$

subject to the constraint. Solve for λ:

$$\lambda = -\frac{I^{-1}a \cdot (I\omega \times \omega)}{I^{-1}a \cdot a}\,.$$

If a is an eigenvector of the moment of inertia tensor flow is measure preserving.

Can extend to Euler-Poincaré-Suslov Equations System is characterized by the Lagrangian $L = \frac{1}{2}\mathbb{I}_{AB}\Omega^A\Omega^B$ and the left-invariant constraint

$$\langle a, \Omega \rangle = a_A \Omega^A = 0. \tag{0.1}$$

Here $a = a_A e^A \in \mathfrak{g}^*$ and $\Omega = \Omega^A e_A$, where e_A, $A = 1, \ldots, k$, is a basis for $\mathfrak{g}$ and e^A is its dual basis. Multiple constraints may be imposed as well. The two classical examples of such systems are the *Chaplygin Sleigh* and the *Suslov problem.* Introduced by Chaplygin in 1895 and Suslov in 1902, respectively.

• Chaplygin Sleigh with Oscillator (work with Marsden and Zenkov)

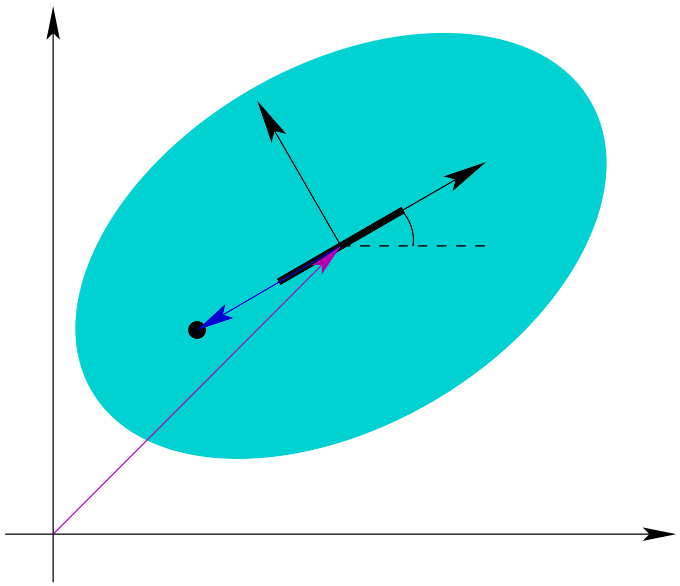

Figure 0.5: Sleigh with Oscillator.

Consider dynamics of the Chaplygin sleigh coupled with an oscillator. We show that the phase flow is integrable, and generic invariant manifolds are two-dimensional tori.

Consider sleigh with a mass sliding along the direction of the blade. The mass is coupled to the sleigh through a spring. One end of the spring is attached to the sleigh at the contact point, the other end is attached to the mass. The spring force is zero when the mass is positioned above the contact point.

The configuration space for this system is $\mathbb{R} \times \mathrm{SE}(2)$. This system has one shape (the distance from the mass to the contact point, r) and three group degrees of freedom.

The reduced Lagrangian $l : T\mathbb{R} \times \mathfrak{se}(2) \to \mathbb{R}$ **is given by the formula**

$$\begin{aligned} l(r, \dot{r}, \xi) = {} & \tfrac{1}{2} m \dot{r}^2 + m \dot{r} \xi^2 \\ & + \tfrac{1}{2} \left((J + mr^2)(\xi^1)^2 + 2mr\xi^1\xi^3 + (M+m)((\xi^2)^2 + (\xi^3)^2) \right) - \tfrac{1}{2} k r^2, \end{aligned}$$

where $\xi = g^{-1}\dot{g} \in \mathfrak{se}(2)$ **and** k **is the spring constant. The constrained reduced energy**

$$\tfrac{1}{2} m \dot{r}^2 + m \dot{r} \xi^2 + \tfrac{1}{2} \left((J + mr^2)(\xi^1)^2 + (M+m)(\xi^2)^2 \right) + \tfrac{1}{2} k r^2$$

is positive-definite, and thus the mass cannot move infinitely far from the sleigh throughout the motion.

Can show the reduced dynamics becomes

$$\frac{Mm}{M+m}\ddot{r} = \frac{Mmr}{(M+m)(J+mr^2)^{\frac{M}{M+m}+1}}\,p_1^2 - kr \tag{0.2}$$

$$\dot{p}_1 = -\frac{mr}{(M+m)(J+mr^2)}\,p_1 p_2, \tag{0.3}$$

$$\dot{p}_2 = \frac{mr}{(J+mr^2)^{\frac{M}{M+m}+1}}\,p_1^2. \tag{0.4}$$

Relative Equilibria of the Sleigh-Mass System:

Assuming $(r, p) = (r_0, p_0)$ **is a relative equilibrium, equation** (0.4) **implies** $r_0 p_1^0 = 0$**. Thus, either** $r_0 = 0$ **and** p_1^0 **is an arbitrary constant, or, using** (0.2)**,** $p_1^0 = 0$ **and** $r_0 = 0$**. Thus the only relative equilibria of the sleigh-mass system are**

$$r = 0, \qquad p = p_0.$$

The Discrete Symmetries and Integrability:

It is straightforward to see that equations (0.2)–(0.4) **are invariant with respect to the following transformations:**

(i) $(r, p_1, p_2) \to (r, -p_1, p_2)$**,**

(ii) $(r, p_1, p_2) \to (-r, p_1, -p_2)$**,**

(iii) $(t, r, p) \to (-t, -r, p)$**,**

(iv) $(t, r, p_1, p_2) \to (-t, r, p_1, -p_2)$**.**

We can now use these transformations to study some of the solutions. We can show $r(-t) = -r(t)$ **and** $p(-t) = p(t)$**.**

$p_1(t) = 0$ **implies that** $p_2(t) =$ **const and that** $r(t)$ **satisfies the equation**

$$\frac{Mm}{M+m}\ddot{r} = -kr,$$

and thus equations (0.2)–(0.4) **have periodic solutions**

$$r(t) = A\cos\omega t + B\sin\omega t, \qquad p_1 = 0, \qquad p_2 = C,$$

where A**,** B**, and** C **are arbitrary constants and** $\omega = \sqrt{k(M+m)/Mm}$**.**

Without loss of generality, we set $A = 0$ **and consider periodic solutions**

$$r(t) = \dot{r}_0/\omega\sin\omega t, \qquad p_1 = 0, \qquad p_2 = p_2^0, \tag{0.5}$$

which correspond to the initial conditions

$$r(0) = 0, \quad \dot{r}(0) = \dot{r}_0, \quad p_1(0) = 0, \quad p_2(0) = p_2^0.$$

We now perturb solutions (0.5) **by setting**

$$r(0) = 0, \quad \dot{r}(0) = \dot{r}_0, \quad p_1(0) = p_1^0, \quad p_2(0) = p_2^0. \tag{0.6}$$

Assuming that p_1^0 **is small and using a continuity argument, there exists** $\tau = \tau_{p,\dot{r}_0} > 0$ **such that**

$$r(\tau_{p,\dot{r}_0}) = 0$$

for solutions subject to initial conditions (0.6)**. That is, the** r**-component is** 2τ**-periodic if** p_1^0 **is sufficiently small.**

Using equation (0.2) **and periodicity of** $r(t)$**, we conclude that** p_1 **is** 2τ**-periodic as well. Equation** (0.3) **then implies that** $p_2(t)$ **is also** 2τ**-periodic.**

Thus, *the reduced dynamics is integrable in an open subset of the reduced phase space*. **The invariant tori are one-dimensional, and the reduced flow is periodic. A generic periodic trajectory in the direct product of the shape and momentum spaces is shown in Figure 0.6.**

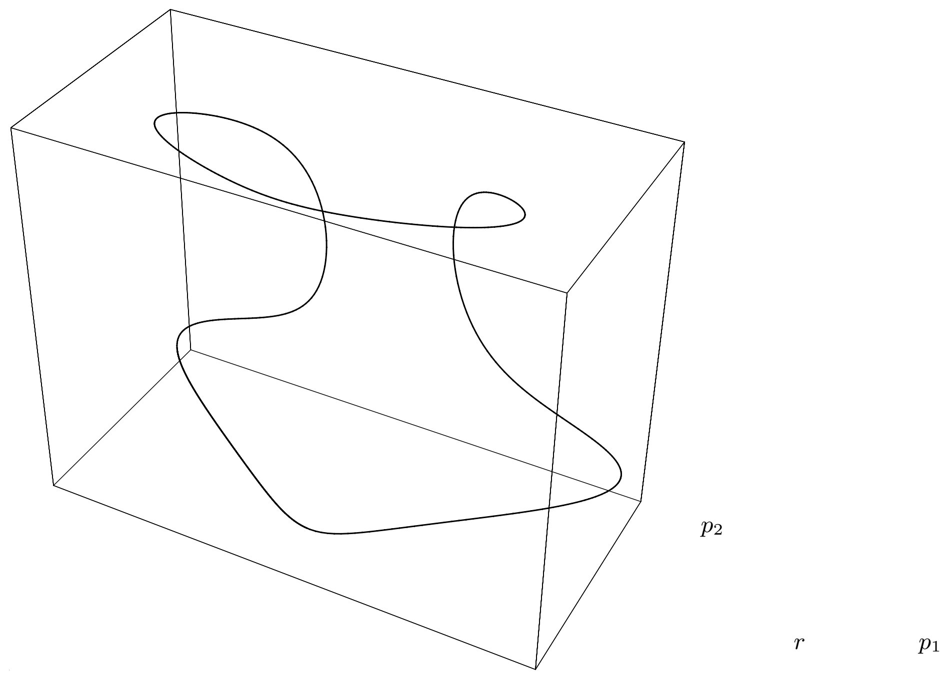

Figure 0.6: A reduced trajectory of the sleigh-mass system.

We obtain the following theorem:

Theorem 0.1 ***Generic trajectories of the coupled sleigh-oscillator system in the full phase space are quasi-periodic motions on two-dimensional invariant tori.***

Typical trajectories of contact point:

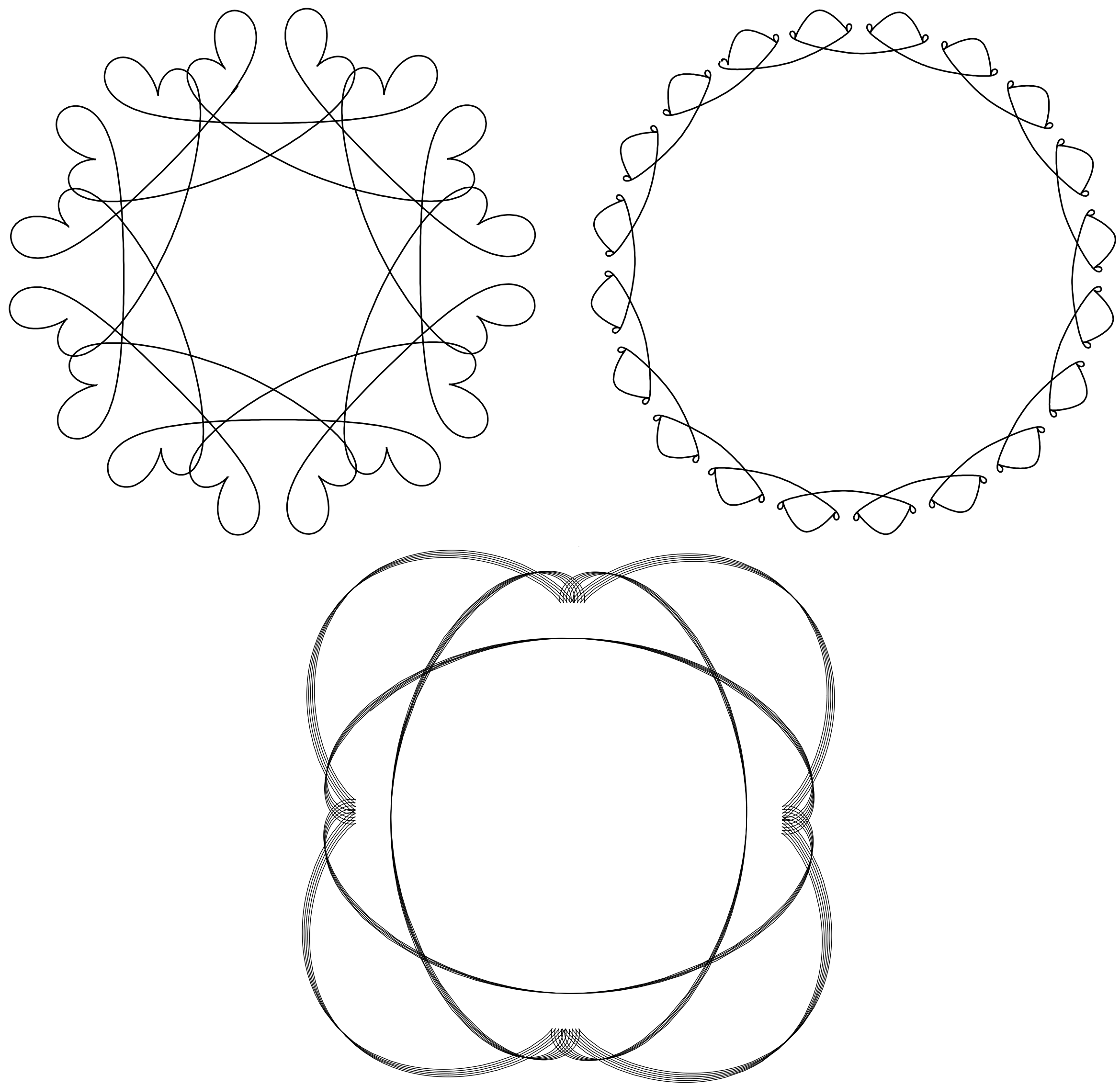

Figure 0.7: Trajectories of the contact point of the blade for various initial states.

•Radiation Damping

See Hagerty, Bloch and Weinstein [1999], [2002].

Important early work: Lamb [1900]. Related recent work may be found in Soffer and Weinstein [1998a,b] [1999] and Kirr and Weinstein [2001].

• Original Lamb model an oscillator is physically coupled to a string. The vibrations of the oscillator transmit waves into the string and are carried off to infinity. Hence the oscillator loses energy and is effectively damped by the string.

• Lamb model

$w(x,t)$ **displacement of the string. with mass density** ρ**, tension** T**. Assuming a singular mass density at** $x = 0$**, we couple dynamics of an oscillator,** q**, of mass** M**:**

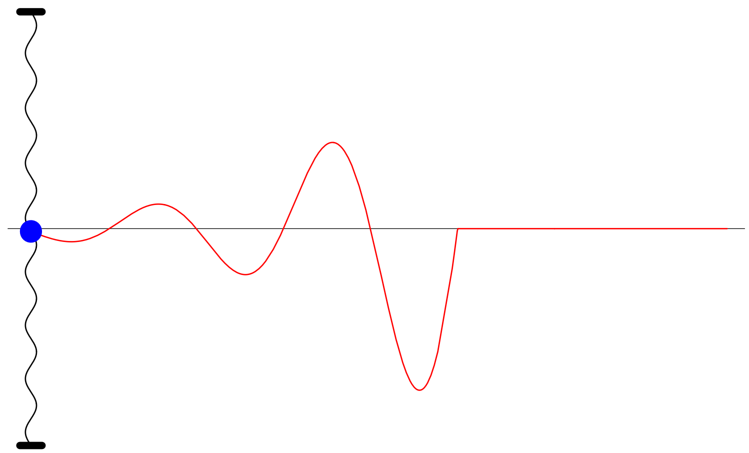

Figure 0.8: **Lamb model of an oscillator coupled to a string.**

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$
$$M\ddot{q} + Vq = T[w_x]_{x=0}$$
$$q(t) = w(0,t).$$

$[w_x]_{x=0} = w_x(0+,t) - w_x(0-,t)$ **is the jump discontinuity of the slope of the string. Note that this is a Hamiltonian system.**

Can solve for w **and reduce:**

• **Obtain a reduced form of the dynamics describing the explicit motion of the oscillator subsystem,**

$$M\ddot{q} + \frac{2T}{c}\dot{q} + Vq = 0.$$

The coupling term arises explicitly as a Rayleigh dissipation term $\frac{2T}{c}\dot{q}$ **in the dynamics of the oscillator.**

• Nonholonomic Systems as Limits

It has been known (see even Cartheodory –1933) that the Lagrange–d'Alembert equations can be obtained by starting with an unconstrained system subject to appropriately chosen dissipative forces, and then letting these forces go to infinity in an appropriate manner.

Kozlov showed that the variational nonholonomic equations too can be obtained as the result of another limiting process: He added a parameter-dependent "inertial term" to the Lagrangian of the constrained system, and then showed that the unconstrained equations approach the variational equations as the parameter approaches infinity.

Nonholonomic constraints can be regarded in some sense as due to "infinite" friction. Several authors have asked if this can be quantified. Interestingly this goes back to the work at least of Caratheodory who asked if the limiting case of such friction could explain the motion of Chaplygin's sleigh. Caratheodory claimed this could not be done but Fufaev (1964) showed that this was indeed possible. The general case was considered by Kozlov (1983) and Karapetyan (1983).

The key idea it so take a nonlinear Rayleigh dissipation function of the form

$$F = -\frac{1}{2}k\sum_{j=1}^{m}\left(\sum_{i=1}^{n}a_i^{(j)}(\mathbf{q})\dot{q}_i\right)^2 \tag{0.7}$$

where $k >$ is a positive constant. Taking the limit as k goes to zero and using Tikhonov's theorem yields the nonholonomic dynamics.

However, the system in this setting is still not Hamiltonian. The goal here is to keep the system in the class of Hamiltonian systems by emulating the dissipation by coupling to an external field

- **The Chaplygin Sleigh (work with Rojo)** **(subRiemannian systems, see work with Brockett, Rojo)**

This system consists of a rigid body moving on two sliding posts and one knife edge, and is perhaps the simplest n.h.s. containing the quasi-dissipative feature mentioned above. This mechanical system has three coordinates, two for the center of mass (x_C, y_C) **and one "internal" angular variable** θ **for the rotation with respect to the knife edge located at** $(x, y) = (x_C + a\cos\theta, y_C + a\sin\theta)$**. The system can rotate freely around** (x, y) **but is only allowed to translate in the direction** $(\cos\theta, \sin\theta)$**: if we choose our coordinates as** $\mathbf{q} = (x, y, \theta)$ **there is a single constraint given by**

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0, \tag{0.8}$$

or, $\mathbf{a}^{(1)} = (\sin\theta, -\cos\theta, 0)$**.**

The equations of motion can be obtained without resorting to the Lagrangian formalism, using simple balance of forces and can be expressed in the form:

$$\begin{aligned} \dot{v} &= a\omega^2, \\ \dot{\omega} &= -\frac{ma}{I + ma^2} v\omega, \end{aligned} \tag{0.9}$$

with $v = \dot{x}\cos\theta + \dot{y}\sin\theta$ **the translational velocity,** $\omega = \dot{\theta}$**,** m **the mass and** I **the moment of inertia with respect to the center of mass.** a **is the distance between the center of mass and the contact point of the knife edge. The solutions of the above equations are ellipses in the** (v, ω) **plane with equilibria given by** $\{v = \mathrm{const}, \omega = 0\}$ **and which are asymptotically stable.**

The above equations can be also obtained using the virtual force method starting with the unconstrained Lagrangian

$$L_0 = \frac{m}{2}\left[\left(\dot{x} - a\dot{\theta}\sin\theta\right)^2 + \left(\dot{y} + a\dot{\theta}\cos\theta\right)^2\right] + \frac{I}{2}\dot{\theta}^2, \qquad (0.10)$$

and using a Lagrange multiplier in the equations of motion:

$$\begin{aligned} m\frac{d}{dt}\left(\dot{x} - a\dot{\theta}\sin\theta\right) &= -\lambda\sin\theta, \\ m\frac{d}{dt}\left(\dot{y} + a\dot{\theta}\cos\theta\right) &= \lambda\cos\theta, \\ (I + ma^2)\ddot{\theta} + ma\dot{\theta}(\dot{x}\cos\theta + \dot{y}\sin\theta) &= 0. \end{aligned} \qquad (0.11)$$

Carathedory and Fufaev added a viscous friction force of the from

$$R = -Nu \tag{0.12}$$

to the sleigh equations.where u is the velocity in the direction perpendicular to the blade. (Note that interchange u and v compared to the original paper of Fufaev.)

Setting

$$k^2 = \frac{m}{I + ma^2}, \quad \epsilon = \frac{I}{Na^2} \tag{0.13}$$

the equations with dissipation become

$$u = \epsilon a\dot{\omega} \tag{0.14}$$

$$\dot{v} = a\omega^2 + \epsilon a\omega\dot{\omega} \tag{0.15}$$

$$ak^2\dot{\omega} + v\omega = -\epsilon a\ddot{\omega} \tag{0.16}$$

It is clear that as ϵ goes to zero one recovers the original equations. Cartheodory incorrectly argued however that since no matter how small ϵ is these equations yield trejectories which

differ from that of the original system, dissipation cannot yield the nonholonomic constraints.

Fufaev realized this is not correct since the system degenerates from a system of three to two equations and thus there is a singularity. Setting $\mu = \epsilon a$ and $\sigma = \dot{\omega}$ we then get

$$\dot{\omega} = \sigma \tag{0.17}$$

$$\dot{v} = a\omega^2 + \mu\omega\sigma \tag{0.18}$$

$$\mu\dot{\sigma} = -ak^2\sigma - v\omega \tag{0.19}$$

Then as $\mu \to 0$ we get rapid motion except for the surface

$$ak^2\sigma + \mu\omega = 0. \tag{0.20}$$

The slow motion of this surface onto the v-ω plane then gives the correct equations of motion.

• **The Chaplygin Sleigh as as Particle in a Radiation Field**

We now show that the sleigh equations can be obtained from a variational principle as reduced equations of motion after the system is coupled to an environment described by an $U(1)$ infinite field of the form $\mathbf{a}(\mathbf{z},t) \equiv [\cos\alpha(\mathbf{z},t), \sin\alpha(\mathbf{z},t)]$. For the Lagrangian of the free field we choose

$$L_{\mathrm{F}} = \frac{K}{2}\int d^2\mathbf{z}\ \dot{\mathbf{a}}^2, \tag{0.21}$$

and we couple the sleigh and the field with a term of the form

$$L_1 = \int d^2\mathbf{z}\ \delta(\mathbf{z}-\mathbf{x})\ \left[\gamma\dot{\mathbf{x}}\cdot\mathbf{a} + \mu\cos\left(\alpha(\mathbf{z},t)-\theta\right)\right]. \tag{0.22}$$

The first term in square brackets corresponds to a minimal coupling that favors ẋ in the direction of a; the second has the form of a potential coupling that favors an alignment of the internal variable θ with the local direction of a.

The total action is $S = \int dt(L_0 + L_F + L_1)$ where L_0 is the Lagrangian of the free sleigh

$$L_0 = \frac{m}{2}\left[\left(\dot{x} - a\dot{\theta}\sin\theta\right)^2 + \left(\dot{y} + a\dot{\theta}\cos\theta\right)^2\right] + \frac{I}{2}\dot{\theta}^2, \tag{0.23}$$

and can be regarded as a full "microscopic" theory of the sleigh coupled to an environment.

The equations of motion of the combined system are now obtained from a variational principle, $\delta S = 0$.

Take the limit $\mu \to \infty$ and use singular perturbation theory.For very large μ we have a very slow dynamics on the r.h.s., which amounts to setting $\sin[\alpha(\mathbf{x}, t) - \theta] = 0$. This is equivalent to saying that in the $\mu \to \infty$ limit the variables $\alpha(\mathbf{x}, t)$ and θ are pinned to the same value. We also obtain

$$\dot{x} \sin \alpha(\mathbf{x}, t) - \dot{y} \cos \alpha(\mathbf{x}, t) = \dot{x} \sin \theta - \dot{y} \cos \theta = 0, \tag{0.24}$$

which means that the constraint is satisfied and one can show the full equations are given also.

Quantum Field Theory

Quantum case for a=0:
The Hamiltonian in this limit has the form

$$H = \frac{1}{2m}\left[p_x - \lambda\cos\alpha(\mathbf{x})\right]^2 + \frac{1}{2m}\left[p_y - \lambda\sin\alpha(\mathbf{x})\right]^2 + \frac{1}{2I}p_\theta^2 \quad (0.25)$$

$$+ \frac{1}{2K}\int d\mathbf{z}\;\Pi^2\left(\alpha(\mathbf{z})\right) + \mu\cos\left[\theta - \alpha(\mathbf{x})\right]. \quad (0.26)$$

For the quantization of H we procede with the usual replacements

$$\mathbf{p} = -i\hbar(\partial_x, \partial_y), \quad p_\theta = -i\hbar\partial_\theta, \quad \Pi\left(\alpha(\mathbf{z})\right) = -i\hbar\partial_{\alpha(\mathbf{z})}. \quad (0.27)$$

For the completely uncoupled case ($\lambda = \mu = 0$) the eigenstates are of the form

$$\Psi_0 = e^{i\int d\mathbf{z}\; m(\mathbf{z})\alpha(\mathbf{z})}e^{i\mathbf{k}\cdot\mathbf{x}}e^{in\theta}, \quad (0.28)$$

with $m(\mathbf{z})$ and n integers and k the wave number of the translational degree of freedom.

The limit $\mu \to \infty$ ammounts to projecting the wave function and the Hamiltonian to states where $\alpha(\mathbf{x}) = \theta$, in such a way that the Hamiltonian becomes

$$H = \frac{1}{2m}\left[p_x - \lambda\cos\theta\right]^2 + \frac{1}{2m}\left[p_y - \lambda\sin\theta\right]^2 + \frac{1}{2I'}p_\theta^2 \tag{0.29}$$

$$+ \frac{1}{2K}\int d\mathbf{z}\,\Pi^2\left(\alpha(\mathbf{z})\right)\left[1 - \delta(\mathbf{x} - \mathbf{z})\right], \tag{0.30}$$

with $1/I' = 1/I + 1/K$. Without loss of generality we can take the quantum numbers $m[(\mathbf{z}) = 0$ for $\mathbf{z} \neq \mathbf{x}$ and the wave function depends only on the $\{\theta, \mathbf{x}\}$ degrees of freedom and obeys the following Schoedinger equation:

$$\left\{\frac{1}{2m}\left[p_x - \lambda\cos\theta\right]^2 + \frac{1}{2m}\left[p_y - \lambda\sin\theta\right]^2 + \frac{1}{2I'}p_\theta^2\right\}\Psi = \epsilon\Psi. \tag{0.31}$$

The above equation can be solved by separation of variables $\Psi = e^{i\mathbf{k}\cdot\mathbf{x}}\psi_{\mathbf{k}}(\theta)$, with $\mathbf{k} = k(\cos\theta_{\mathbf{k}}, \sin\theta_{\mathbf{k}})$ a quasi-translational wave-

vector. The reduced equation satisfied the the angular part of the wave function is

$$\left\{\frac{1}{2I'}p_\theta^2 - \frac{\lambda\hbar k}{m}\cos(\theta-\theta_{\mathbf{k}})\right\}\psi_{\mathbf{k}}(\theta) = \epsilon'\psi_{\mathbf{k}}(\theta), \tag{0.32}$$

with $\epsilon' = \epsilon - (\lambda^2 + \hbar^2 k^2)/2m$**. This equation has well known solutions in terms of the Mathieu functions. One can gain insight on the structure of the solutions by looking at the fast limit** $(k \to \infty)$ **which should exhibit features of the classical solution. In this limit the fluctiations of the angle are small and centered around** $\theta = \theta_{\mathbf{k}}$**. This means that, up to small quantum fluctuations, the knife edge is pointing in the direction of the plane wave propagation. Expanding for small values of the angle we find that the solutions in the fast limit are of the form**

$$\Psi_{\mathbf{k}}(\mathbf{x},\theta) = e^{i\mathbf{k}\cdot\mathbf{x}}e^{-(\theta-\theta_{\mathbf{k}})^2/2\Delta_\theta^2}, \tag{0.33}$$

with

$$\Delta_\theta^2 = \frac{m\hbar}{\lambda k I'}. \tag{0.34}$$

Some key refs:

Nonholonomic Mechanics and Control, **Springer Graduate Text, 2 003 (with the colloboration of J.Baillieul, P.E. Crouch and J.E. Marsden).**

Radiation Induced Instability, *SIAM J. Applied. Math.* **64 (2004), 484-524 (with P. Hagerty and M. Weinstein).**

Invariant Measures of Nonholonomic Flows with Internal Degrees of Freedom, *Nonlinearity* **16 (2003) , 1793-1807 (with D. Zenkov).**

Quantum subRiemannian dynamics, in it New Directions and Applications in Control Theory, ed. W.P. Dayawansa, A. Lindquist and Y. Zhou, Springer Lecture Notes in Control and Information Sciences, 2005 (with R. Brockett and A. Rojo)

Quantization of a nonholonomic system, *Physical Review Letters* **101, 030404, 2008. (with A. Rojo).**

Hamiltonization of Nonholonomic Systems and the Inverse Problem of the Calculus of Variations, to appear in *Reports on Math-*

ematical Phsyics **(with O. Fernandez and T. Mestdag).**
Quasivelocities and Symmetries in Nonholonomic Systems, to appear in *Dynamical Systems* **(with J. Marsden and D. Zenkov).**