The Differential Analyzer as an Active Mathematical Instrument

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Introduction

This presentation features the essential role played by amplification and control in the successful development of the differential analyzer (DA). This first occurred in 1930 with the introduction of the capstan torque amplifier in Vannevar Bush’s largely mechanical form and then later, in 1947, with the use of the voltage operational amplifier in an electronic embodiment that was called an electronic analog computer. Certain essential components and features of both machines are outlined and contrasted, where the fast all-electronic machine best exemplifies the distinctive benefits attending the evolution from mostly mechanical to fully electrical implementation.

Background

In 1931, Professor Vannevar Bush of the Massachusetts Institute of Technology (MIT) announced to the world the existence of a newly constructed machine for solving ordinary differential equations (ODEs), which had been named the “differential analyser” (by Waldo Lyon) [1]. This instrument was the culmination of six years of computing machine construction and use by Bush and his several collaborators, including most notably Herbert Stewart, Harold Hazen, and Samuel Caldwell [2]-[4]. However, the concept of this machine was not in itself new but had, in fact, been promulgated a half century earlier by Sir William Thomson (soon to become Lord Kelvin) [5], [6]. At that time, Kelvin and his older brother, James Thomson, had similarly constructed a series of ingenious mathematical engines for the purpose of tidal harmonic analysis and synthesis [7]-[9]. But Kelvin did not construct such a differential analyzer, not for lack of precision machining in his time, but rather because, like Charles Babbage before him, he failed to appreciate the value of isolation between components through use of power amplifiers.

Thus it was precisely this essential contribution of the Bush team that assured 30 years of analog supremacy in scientific and engineering computation, terminated only by the ultimate impact of integrated circuitry and user-oriented software upon digital computing. This use of control technology serves as the theme of this lecture.

From Planimeters to Integrators

The art of analog devices was active in Kelvin’s time, particularly that which concerned the graphical determination of definite and indefinite integrals. The planimeter was an instrument used to determine the area bounded by a closed curve as on a map. In contrast, an integrator was used to determine the running integral of a plotted curve. In 1855, James Clerk Maxwell (the Maxwell?), who had seen Amsler’s and other planimeters, was convinced that the combined rolling and slipping limited the accuracy of the instrument. In response, he devised a new planimeter based on the mutual rolling of two equal spheres [10]. Maxwell’s criticism led directly to James Thomson’s simplification in the form of a disk-ball-cylinder integrator operating by pure rolling [11]. This same idea later inspired Hannibal Ford’s 1920 invention of the disk-double-cylinder integrator [12].

By 1876, in part from his long experience at sea in connection with his submarine cable telegraph activities, Kelvin had become much interested in tidal analysis and prediction, particularly in the task of extracting the lunisolar harmonic coefficients from tidal
Kelvin's successful implementation of the above mathematical instruments for tidal analysis and prediction led him to comment that such machines substituted "brass for brain in the great mechanical labor of calculating the elementary constituents of the whole tidal rise and fall." His enthusiasm led him soon to make the discovery that an interconnected chain of his brother's integrators could be used to solve ODEs directly. At first, this was accomplished by simple iteration, in a Picard manner identical to that later employed by Bush, Hazen, and others with their early integrators [3]. But finally, in Kelvin's words: "So far I had gone and was satisfied, feeling I had done what I wished to do for many years. But then came a pleasing surprise. Compel agreement between the function fed into the... machine and that given out by it." [5]

This was to be done by establishing a mechanical feedback connection causing the integrator of the first integrator in the chain to be in accord with the functional constraints imposed by the outputs of all subsequent integrators. The resulting closed-loop motion of the machine would necessarily yield the solution.

Although the scheme outlined by Kelvin in this and a subsequent paper [6] is exactly that familiar to all of us today, Kelvin did not go on to say precisely how to compel such agreement nor what kind of connection to establish. So a half century had to elapse for the Bush team to make the appropriate connection through a servo amplifier to compel successfully the desired agreement. We shall report more on this shortly.

Activation via Amplifiers

Let us now consider the emergence of amplifiers and their use in mathematical instruments and other analog components.

A purely passive instrument is one with no source of additional energy between input and output. For such a device, all work done by the output must come from the input itself, just as with hand manipulation of the planimeter stylus. Yet accuracy will diminish in direct proportion to the resultant load inside and outside the instrument. It follows that if we wish to use the output of one device to drive others, we must attend to power and energy demands.

A power amplifier is any device whose output depends on an input signal yet which draws most of its power from a source other than the input signal. Thus the three essential parts of any amplifier are the input connection(s), the output connection(s), and the primary source of energy. In the simplest case, these will thus constitute three ports.

In this event, amplifying devices may be more simply considered as active two ports possessing an internal embedded energy source controlled by the low-power input signal at one port and capable of driving a higher-power load at the other port. Using this image, the Table outlines the distinction between passive and active mathematical instruments in both mechanical and electrical forms.

This distinction between passive and active two-port devices is clarified by the four panels of Fig. 1 (an adaptation from Shockley [13]).

**Mechanical Transformer** The upper-left panel represents the simplest and earliest transformer: the mechanical lever or crowbar, whose first application is prehistoric and even prehuman. We all know now that this device transforms a small force at one end into a larger force at the opposite end. But this "amplification" occurs only with a corresponding reduction in the distance moved, in direct consequence of energy conservation. This property holds for all two-port passive mechanisms.

**Electrical Transformer** The lower-left panel depicts the analogous electrical transformer emergent in the 1850s after the discovery of mutual induction by Michael Faraday and Joseph Henry. As before, a step-up transformer can increase the voltage only at the expense of current: the output power cannot exceed the input power.

**Mechanical Amplifier** A comparably ancient mechanical amplifier is the capstan device in the upper-right panel, well-known to early seamen in a primitive form. Here the energy source is the drive motor, which continually rotates the drum around which a rope is wrapped. As a result of the exponential increase in rope tension due to Coulomb friction, a light pull at one end of the rope will now move a heavy load to an equal distance. The capstan thus acts as a simple servo follower to produce a replica of the input motion but now with a hundredfold increase in force, the necessary power augmentation coming from the drive motor. A reversible and stageable form of this remarkable torque amplifier was patented by Henry W. Nieman [14], [15] and played a critical part in the first mechanical differential analyzer (MDA), as discussed subsequently.

**Electrical Amplifier** Once again a direct electrical analog is shown below in the form of the now-familiar NPN emitter-follower available by the mid-1920s and which was, in turn, patterned after the earlier cathode-follower of the 1920s. Any voltage variation at the base is now tracked closely at the emitter but with a typical power gain of 100 or more; this increase, of course, coming from the collector supply. However, this ingenious and revolutionary device was not widely available until this later time and could not figure in the birth of the MDA or electronic differential analyzer (EDA).

From the preceding discussion, it becomes clear that two quite distinct alternatives exist...
for the particular case of MDAs employing rotary mechanical integrators:

1. A passive embodiment wherein each integrator is constructed ruggedly enough to develop torque adequate to drive several subsequent interconnected integrators and other devices. This was successfully accomplished by Hannibal Ford during World War I [12].

2. An active embodiment wherein each integrator is isolated via a servo amplifier such that adequate power is extracted from a separate source to drive any number of other like devices. This was the accomplishment of Bush and his colleagues [1].

In the following, we consider only the active form of such devices, even though there is evidence that purely passive machines to solve certain low-order ODES were constructed in Spain, Russia, and Germany prior to the efforts at MIT.

The First Mechanical Differential Analyzer

In 1928, Bush and Hazen commenced to build a six-integrator DA good to three-figure accuracy. At least for the simplest linear and nonlinear systems, the machine was to be fully automatic, but as in the earlier machines it should also be capable of receiving empirical input data.

Disk-and-Wheel Integrators

Back when an additional second-stage integrator was to be added to the original watthour-meter product integrator [2], they chose a simple disk-and-wheel integrator [3], as shown in Fig. 2. Because they had gained much experience and familiarity with this device, it was decided to build the first DA using such integrators. However, because they were now striving for an order-of-magnitude improvement in accuracy, obtained by using a wheel with a finely sharpened edge set in jeweled bearings, making near-point contact with the hardened-steel disk, the output could drive very little load and could only control rotation. The necessary work would have to come from some auxiliary source in the form of an amplifier. The Bush group then sought a suitable isolating amplifier.

Nieman Torque Amplifier

By a fortunate coincidence, Henry Nieman of Bethlehem Steel had recently announced [14] his new torque amplifier, one variant of which worked with counterrotating capstans equipped with wrapped cords (or bands) passing 2.5 turns around the drums, as shown in Fig. 3. The low-power input shaft pulls at one end of the band, which rubs on the continually revolving drum as it passes around it, and finally pulls the output shaft along at the same speed as the input but now with much increased torque and power. The two motor-driven drums provide for both directions of rotation. If two stages were used, a power gain as high as 10,000 could be achieved. The final design then employed such two-stage torque amplifiers directly attached to the output shafts of the integrators.

Function Generation and Input Tables

Functions were introduced via their defining algebra or analysis or otherwise introduced graphically via an input table manipulated by a skillful human operator.

Output Tables and Counters

Intermediate and final outputs could be recorded either graphically on output tables or read numerically from revolution counters, which registered the shaft angles.

Final Machine Arrangement

As the adders and integrators each had two inputs and one output and the machine could be envisioned as a chain of six integrators closed through a feedback coupling function, the dependent and independent variables all could be carried on 18 tubular “bus” shafts running the length of the machine. The integrator/torque-amplifier sets were located on one side of the machine, and the input and output tables and multiplier/dividers were on the opposite side. These devices were coupled through cross-shafts via right-hand (+) or left-hand (−) helical gears to the longitudinal bus shafts. (Self-explanatory schematics expedited the setup.)

![Fig. 2. Disk-and-wheel integrator.](image)

![Fig. 3. MIT form of Nieman torque amplifier.](image)
The completed machine and samples of its use were described by Bush in [1].

Clones of the MIT Differential Analyzer

With the original MIT MDA blazing a trail, some large and small copies, or "clones," soon appeared. First, two 10-integrator machines were built patterned after the MIT machine as part of a joint program begun in 1933 between MIT, Army Ordnance, and the University of Pennsylvania Moore School. When completed, one of these machines was located at the Ballistics Research Lab in Aberdeen, Maryland, and the other at the Moore School.

Then prior to World War II, two eight-integrator machines were assembled at Manchester and Cambridge Universities in the United Kingdom. A 12-integrator MDA was built in Oslo, Norway, and a six-integrator clone at Leningrad, USSR. However, in 1943 General Electric (GE) completed an entirely new 14-integrator MDA [1] using polarized-light servos [17] instead of capstan amplifiers to isolate each integrator wheel from the output load. A close duplicate of this unit was built by GE shortly after this and installed at the University of California at Los Angeles.

The Meccano Clones

Perhaps the most notable of all the MDA clones were the three built early in the United Kingdom for feasibility studies constructed from children's "Meccano" sets (similar to the U.S. Erector sets) at Manchester [18], Cambridge [19], and Belfast [20]. Nearly every part came from the set except for the plate-glass surfaces clipped to the integrator disks. What most surprised was the consistent 1–2-percent accuracy obtainable even with such rudimentary devices. In each case, two-stage Nieman-type torque amplifiers were built from the standard Meccano parts supplied. Surely every single part could have been fabricated in Kelvin's time; he lacked only this concept of amplification.

The Rockefeller Differential Analyzer

The success of the first MIT machine not only inspired the earlier clone-builders, but also prompted the MIT group to undertake development of a more elaborate and improved DA based on the collective experience of all concerned with the first-generation machines. This program was begun in 1935 with the new machine placed into secret wartime service in 1942, under the direction of Dr. Caldwell. Its existence was finally announced to the public in a 1945 paper by Bush and Caldwell [4].

Both old and new MIT machines operated on the same principles and their essential computing mechanisms were the same. But a primary objective of the new design was to improve precision. This was achieved to the extent that each component of the new machine delivered results to within one part in 10,000 (or 0.01-percent precision equivalent to four decimal digits), whereas corresponding parts of the first machine gave little better than 0.1-percent precision. This design standard resulted in a final machine having an overall accuracy better than 1 in 1000 and a typical solution time of 15 min. Indeed, this perhaps was the most accurate large-scale analog computer ever constructed!

This new DA had 18 integrators available, three times the number in the first model and four more than in the next largest machine then in use. The principal distinction of the RDA (Rockefeller Differential Analyzer) over all earlier models was the ease and speed with which it could be set up for the solution of complex problems. This was enabled through a two-wire data-transmission system employing standard telephone crossbar switches and relays, which were used for all interconnection purposes. Instead of the earlier torque amplifiers, capacitive synchro angle resolvers isolated all components and produced these two-wire signals. Punched paper tapes for machine setup then allowed simple, rapid connection (and reconnection) of the DA.

The larger number of available integrators, together with associated input/output (I/O) equipment and general ease of use, now encouraged the solution of very complex problems, including the analysis and synthesis of nonlinear control systems of various types. However, by the end of the war in 1945, MDAs had already reached the peak of their popularity, with no more than a few dozen available throughout the globe and with fewer than 200 integrators in all. They were, in fact, about to be eclipsed by the appearance of the EDA and relegated to museums.

Genesis of the Electronic Differential Analyzer

In 1947, John Ragazzini and his colleagues at Columbia University announced to the world the existence of a yet newer machine that solved dynamics problems via active circuits [21]. This device incorporated what were called operational amplifiers as the activating elements, playing for integrating circuits a role directly comparable to the torque amplifiers in the first MDA.

This work had, in fact, been carried out under a wartime contract administered by Section 7.2 (Airborne Fire Control) of the National Defense Research Council (NDRC), whose Chief was none other than Samuel Caldwell and whose supervising Technical Aides were J. B. Russell (also a Columbia Professor) and George A. Philbrick (formerly of Foxboro Company). Directing this group were Harold Hazen as Chief of the entire Section 7 (Gunnery Fire Control) and heading all of NDRC, Van Bush himself.

This same NDRC section had commissioned the Bell Labs work for the M9 anti-aircraft gun director, whose amplifier circuit was the immediate progenitor of the Ragazzini amplifier. Thus, we note that there existed an unbroken path from the original MDA to the ultimate EDA.

Operational Amplifiers

Although Philbrick had pioneered in the use of computing with voltages via active circuits at the Foxboro Company as early as 1938, his lasting influence began with his NDRC activities after 1942 and, of course, with his launching of his company (G. A. Philbrick Researches, Inc., or GAP/R) in 1947 after the war. As a result, he shares with Lovell the credit for the invention of the operational amplifier, which has become our latter-day integrated circuit operational amplifier or "op amp" [22], [23].

During World War II, the principal driving force to move from purely mechanical to electromechanical and finally to purely electronic devices was the urgent need to increase speed while reducing weight. This requirement was particularly acute for interfacing with airborne radar systems, where the sensors, amplifiers, and actuators were already electronic. The name "operational amplifier" for the resulting high-gain direct-current (DC) device alludes to its readily ability to provide any desired transfer function by simple activation of resistor-capacitor circuitry. Not only were all heavy precision mechanical parts thus avoided, but also even the familiar bulky inductor coils and transformer couplings, with their tendency for saturation and cross talk.

From the outset, the op amp had, at minimum, the essential properties of high-gain, high-input impedance, low-output impedance, wide bandwidth, low drift, and low noise. The stringency of these requirements delayed the introduction of the EDA until after the end of World War II.
The ultimate postwar impact of these devices then rested on the fact that, in contrast to the MDA, they were light, small, fast, versatile, and readily assembled from standard radio and TV parts. By this time, Radio Shack and Heathkit had replaced Gilbert Etector, British Meccano, and German Marklin.

Most early operational amplifiers, like Lovell’s M9 circuit, were DC amplifiers having a single inverting voltage input. But Philbrick departed radically from this tradition and began shortly after 1950 to use and market separately true differential operational amplifiers such as the K2W with plug-in packaging. This remarkable groundbreaking device exploited the thermal symmetry of certain 12AX7 twin triodes (notably Tungsol), which gave excellent common mode rejection when used differentially as long as the heaters were left on. This, together with some other clever internal features (such as the radium-painted neon stabilizing bulbs), led to this plug-in K2W differential operational amplifier becoming the “Model T” of analog computing!

Chopper Stabilization

Also by 1950, Goldberg and Williams had independently published designs using vibrating switches, or “choppers,” for drift stabilization of amplifiers [24], [25]. We outline below an implementation in terms of Philbrick components. This crucial step now enabled the relatively inexpensive EDA to match the accuracy of even the very best and most expensive MDA.

Although most of the dozen or more EDA manufacturers continued to manufacture single-input but now stabilized amplifiers, Philbrick could take the unique alternative step of providing a separate alternating-current stabilizing amplifier (the K2P), whose output now operated through the positive non-inverting input of the K2W. The low-cost chopper was that commonly employed in car radios (before transistors) to invert battery voltage to the higher plate supply; even so, this small chopper alone cost more than the K2W op amp!

This separate packaging meant that only those who needed the lower drift rate and resulting higher gain would incur the increased cost. For most applications, the inexpensive K2W alone sufficed. Again, like Henry Ford’s Model T in an earlier era, Philbrick’s Model K2W helped spread yet another “new-fangled contraption” over the planet.

Breadboard Computing

Because these Philbrick products catered to the “do-it-yourselfers,” many active circuit aids became available. Notable among these were the first tabletop or rack-mounted op-amp manifolds—veritable analog “personal computers”—which soon were copied and became standard. These are still made by Comdyna and others and, of course, now employ solid-state op amps.

The smallest such totally self-contained unit in the earlier tube era was the Model MK. It could hold four K2Ws or two K2W-K2P pairs. However, more than half the volume was necessarily occupied by the two bulky, heavy, and costly ±300-V tracking power supplies.

EDA Mainframes

More conventional EDAs, such as the top-of-the-line units by Electronic Associates, Inc. (EAI), used patchcords and plugboards for interconnections. However, Philbrick mainframe computers started out instead with black boxes interlaced by cables and ended up 15 years later with entirely switch-programmed machines.

On a global basis, by 1965 many millions of operational amplifiers had been made and sold. Even single installations had more computing power than all previous MDAs combined. Yet by this time the all-analog EDAs were obsolete and hybrid mainframes were becoming increasingly common. Only EAI has remained in this line of business. Now for dynamic analysis even the cheapest desktop and laptop personal computers can outperform all but the largest and/or fastest of the EDAs.

EDA I/O Components

The EDAs required I/O components equivalent to those of the original MDA. But now for high-speed machines these necessarily had to be all-electronic. We now consider some of these devices.

Shopper/Auctioneer Logic

These are the analog equivalents of binary diode logic. Out of compound active nets of these primitives, essential EDA I/O and nonlinear elements could be constructed.

Electronic Graph Paper

Before the advent of modern video displays, Philbrick used such logic circuits to produce calibrated displays.

Multiplier/Dividers

As in the MDA, these units were required for algebra. But as the capacitor-integrators could do only time integration, multipliers were also needed for integration with respect to arbitrary variables. For this need, Philbrick developed, among other variants, a carrier-modulated diode circuit.

Function Generation

The role of the input tables of the MDAs must now be provided by electronic function generators. Based diode-function generators for a single variable had become common, but Philbrick introduced two innovations. Again by using triangular carrier modulation, tangent parabolas could smooth the curves between line segments. For functions of two or more variables, diode selection logic produced a general class of piecewise linear functions. A typical geometry could be generated by the Philbrick function of two variables; once more, the sharp points could be ground off by either modulation.

The preceding battery of equipment was used not only in many mainframe EDAs, but it also became common laboratory instrumentation.

Conclusion

Somewhat reluctantly, with the advent of the electronic operational amplifier and then the integrated circuit op amp, the mechanical differential analyzer has long vanished from the scene. It was then soon followed by the extinction of the slide rule itself—that paragon of analog computing—brought on by the dominance of the transistorized pocket calculator!

Yet the MIT mechanical differential analyzer represented a truly significant pioneering effort in machine computation. Perhaps more than any other single device, it served to convince scientists and engineers that mathematics by machine was both practical and useful, thereby spurring on the later rapid development of electronic analog and digital computers. Certainly the enormous World War II influence of the team of Bush, Hazen, Caldwell, and Philbrick furthered this cause.

But far less well-recognized has been the clear and self-evident demonstration of the extraordinary contributions to scientific and
technical progress engendered by amplification and control concepts. This is the true heritage of the Bush achievement.

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References


Out of Control

"He says he was merely going to inject a disturbance in the plant to measure the return difference matrix!"