ABSTRACT: Eigenstructure assignment for linear systems is basically an inverse eigenvalue problem. Two possible solutions to this problem are presented here that have been implemented on the Ctrl-C® Commercial Design Package. Both algorithms use singular-value decomposition and produce robust (well-conditioned) answers to the state-feedback pole-assignment problem. An example is included to assess the relative strengths of the two techniques.

Introduction

Eigenstructure assignment is a procedure used to develop a linear gain matrix for feedback control so that the resulting system has a desirable closed-loop response. Moore [1] has given necessary and sufficient conditions for such a gain matrix to exist. When knowledge is at hand as to a desired shape for the eigenvalues, it is from these subspaces that the best choice (in whatever sense) for each eigenvector is made.

A full-force nonlinear model of an aircraft is the test system employed, and, since the modal assignment is known to be well-conditioned, this is used as a "control" to assess the viability of the iterative scheme.

Eigenstructure Assignment

Consider a system modeled by the following linear, time-invariant, state-space representation, where \( x \) is the \( (n \times 1) \) state vector, \( u \) the \( (m \times 1) \) control vector, and \( y \) the \( (p \times 1) \) output vector.

\[
\begin{align*}
x &= Ax + Bu \\
y &= Cx
\end{align*}
\]

The \((A, B, C)\) triple is assumed to be both controllable and observable. It is assumed, without loss of generality, that \( \text{rank}(B) = m \) and \( \text{rank}(C) = p \). We assume here, as in the case of many problems, that the number of outputs exceeds the number of inputs. If no plant commands exist, the control vector simply consists of a real matrix times the output vector \( y \).

\[
u = Ky (= KCx)
\]

The eigenstructure assignment problem may be formulated as follows [4]: Given a set of self-conjugate desired eigenvalues \( \{\lambda_{i}\}, i = 1, 2, \ldots, p \), and a corresponding set of (self-conjugate) desired \( n \) vectors \( \{v_{i}\}, i = 1, 2, \ldots, p \), compute a real \( (n \times p) \) matrix \( K \) such that \( p \) of the eigenvalues of the closed-loop matrix \((A + BK)\) are precisely those of the set \( \{\lambda_{i}\} \), with the \( p \) corresponding eigenvectors as close as possible to the set \( \{v_{i}\} \).

The feedback matrix will exactly assign those \( p \) desired eigenvalues. A corresponding desired eigenvector may be assigned exactly only if it lies in the subspace spanned by the columns of the first \( n \) rows of the matrix representing the kernel (or null space) of \((A - \lambda_{i}I)B\) [6], [7]. The dimension of this subspace is \( m \) (the number of independent control variables). In general, it is not possible to assign a particular desired eigenvector if the number of specified elements is greater than \( m \) [12]. If this is the case, the desired eigenvector is projected in a least-squares sense into the allowable subspace [4].

The first technique to be discussed is direct eigenstructure assignment. In this instance, a desired set of eigenvalues and eigenvectors is prespecified by consideration of the required closed-loop response. The desired eigenvalues may be achieved precisely and the best-fit eigenvectors are assigned. If a set of closed-loop eigenvectors of specific form cannot be deduced or is indeed unnecessary (e.g., in a chemical plant), the eigenvectors may be constrained in such a way that the eigenvalues are insensitive to perturbations in the system matrices.

Wilkinson [8] showed that the sensitivity of the eigenvalues of a matrix to perturbations decreases as the orthogonality of the corresponding eigenvectors increases. This is the basis for the second technique. Kautsky et al. [9] were the first to utilize such eigenvector orthogonality information in their algorithm, which was implemented on the MATLAB package (by Cleve Moler, University of New Mexico). The algorithm they used is an iterative procedure that involves the replacement of each closed-loop eigenvector, in turn, by a new one, taken from the allowable subspace, which is maximally orthogonal to the space spanned by the remaining eigenvectors. However, the algorithm had problems with complex poles, which were later overcome in a modified version developed by Mudge and Patton [6], [7]. The algorithm implemented on Ctrl-C by the authors is a further variant of this latter technique producing comparable results. This robust eigenstructure assignment algorithm is simple to implement but is not guaranteed in all cases to converge to the optimum assignment, although considerable improvement in conditioning is always possible. An alternative to the iterative approach is the direct assignment of a set of orthogonal vectors, which, if perfectly matched, gives optimal conditioning. If the open-loop \( A \) matrix is fairly well conditioned, then taking the nearest orthogonal matrix to its matrix of eigenvectors would be a reasonable choice for the set of desired eigenvectors. It is apparent that this approach helps to minimize
the eigenvectors of the closed-loop matrix will be similar to the open-loop ones.

Kautsky et al. [9] also provide a further method of determining for a chosen set of closed-loop eigenvalues if it is possible to assign the corresponding eigenvectors to yield a well-conditioned closed-loop matrix. This method is particularly useful when there is some freedom in the placement of some (or all) of the closed-loop eigenvalues. An optimization process (e.g., method of steepest descent) may be used to find the best set of desired eigenvalues from those available; this is the focus of some current work by the authors.

The method used for computation of the linear gain matrix \( K \) [9] in both algorithms depends on the conditioning of the eigenproblem of the closed-loop matrix \((A + BK)\). If the conditioning is poor, then the computation of \( K \) may be inaccurate [13]. Normally, this would not be a problem because the aim of the robust eigenstructure assignment is to produce a well-conditioned eigenproblem for the closed-loop matrix! Exceptionally, if only a badly conditioned set of acceptable closed-loop eigenvalues can be found, the method of Minimis and Paige [13] is recommended.

**Eigenstructure Assignment Package**

Ctrl-C is a computer-aided engineering workbench for matrix analysis, digital signal processing, engineering graphics, and, in particular, control system design and analysis [14]. Therefore, the authors felt it was necessary to implement eigenstructure assignment routines in order to enhance and develop the library of functions available. It is hoped to improve the software written thus far and, in the future, produce a commercially viable version for other Ctrl-C users and to make a contribution to packages such as ECSTASY (Environment for Control System Theory, Analysis, and Synthesis [15]).

The package has been written in the Ctrl-C programming language using DO procedures and user-defined function capabilities available within it. The algorithms described in the text have been implemented with a number of other facilities incorporated. If necessary, the controllability and observability of the \((A, B, C)\) triple may be checked, and any uncontrollable (invariant) eigenvalues may be included in the set of assigned eigenvalues with corresponding extra degrees of freedom to assign the associated eigenvectors [16]. The misalignment angle between the desired and achieved eigenvector is displayed so that the user has a feeling for how good a fit has been realized. Open-loop poles may be assigned also, because there is no need to calculate the inverse of the singular matrix \((A - \lambda I)\) with the algorithm portrayed here.

The designer may wish to assign multiple eigenvalues with multiplicities larger than \(m\). Since only \(m\) independent eigenvectors may be assigned, generalized eigenvectors must be produced. This is possible [11] with the use of SVD; this capability recently has been built into the direct assignment section of the package with an appropriate modification to the iterative design currently under consideration.

**Aircraft Example**

The example employed here is the lateral motion of a light aircraft [5]-[7]. The equations that describe the aircraft in flight are nonlinear; therefore, a suitable linearization about a nominal state trajectory is performed. This “stick-fixed” linearization yields the model \( \dot{x} = Ax + Bu, y = Cx = x \), where \( x \) is a seven-dimensional state vector with the following components: sideslip velocity \((v)\), roll rate \((p)\), yaw rate \((r)\), roll angle \((\phi)\), yaw angle \((\psi)\), rudder angle \((\gamma)\), and aileron angle \((\xi)\). \( u \) is a two-dimensional control vector with the elements rudder angle demand \((\xi_c)\) and aileron angle demand \((\xi_a)\), respectively.

The state matrices corresponding to a 33-msec\(^{-1}\) airspeed are given in Table 1. This system has the output vector exactly equal to the state vector, and the \((A, B)\) pair is completely controllable so that all seven closed-loop eigenvalues may be specified. The aircraft problem has been chosen because an ideal eigenstructure may be readily proposed [6], [7], [17]. The lateral motion is characterized by three modes: roll, dutch roll, and spiral. The eigenvalues are chosen for each mode so that the necessary damping requirements are fulfilled, with the corresponding eigenvectors selected to produce the necessary decouplings in order to yield suitable handling qualities. For instance, the dutch-roll mode should have a damping ratio of about 0.7, and the spiral mode is associated with the roll angle only and should not be coupled into sideslip velocity (so as to avoid sideslip in steady turns); hence, a zero element is specified as the first entry in the spiral mode eigenvector. The desired eigenstructure is as follows:

\[
\begin{align*}
\lambda_1, \lambda_2 &= -2.0 \pm j1.0 & (\text{roll mode}) \\
\lambda_3, \lambda_4 &= 1.5 \pm j1.5 & (\text{dutch-roll mode}) \\
\lambda_5 &= -0.05 & (\text{spiral mode}) \\
\lambda_6 &= -15 & (\text{rudder actuator mode}) \\
\lambda_7 &= -10 & (\text{aileron actuator mode}) \\
\end{align*}
\]

\((* = \text{unspecified element}) (4)\)

**Table 1**

**State Matrices**

\[
A = \begin{bmatrix}
-0.277 & 0.000 & -32.900 & 9.810 & 0.000 & -5.432 & 0.000 \\
-0.103 & -8.325 & 3.750 & 0.000 & 0.000 & 0.000 & -28.640 \\
0.365 & 0.000 & -0.639 & 0.000 & 0.000 & -9.490 & 0.000 \\
0.000 & 1.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & -10.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & -15.000 \\
\end{bmatrix}
\]

\[
B^T = \begin{bmatrix}
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 20.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 10.000 & 0.000 \\
\end{bmatrix}
\]
The modal matrix of eigenvectors was formed using direct eigenstructure assignment [6], [7]. Then a second matrix was formed using direct eigenstructure assignment 161, 171. Then a second matrix was produced using direct eigenstructure assignment. Both procedures are illustrated in Table 2. The torr. and aileron actuator eigenvectors are all italic format represents the elements that are specified for the direct assignment.

As is apparent, by inspection of DIR in Table 2, the dutch-roll, spiral, rudder actuator, and aileron actuator eigenvectors are all achieved exactly. The desired eigenvector for the roll mode did not reside in the allowable subspace; hence, the least-squares approximation has been given. The misalignment angle between the desired and achieved eigenvectors is 16.1 deg, indicating that an eigenvector close to the desired one was assigned.

Inspection of the Matrix IT in Table 2 reveals a close fit between the desired eigenvectors (for the direct assignment) and the assigned eigenvectors of the iterative approach for the roll, dutch-roll, and spiral modes. This surprising result could not be expected, in general, because desired modal response comes only from aircraft handling-quality considerations. It could be argued, however, that in decoupling the modes of the system using the direct assignment method, the eigenvectors are assigned, to some extent, to be orthogonal. This is precisely the aim of the iterative algorithm. Therefore, the configuration of the closed-loop eigenvectors realized by the iterative assignment should be expected, sometimes, to be comparable to that of the direct (modal) assignment.

Using the approach of Wilkinson [8], the sensitivity, c_i, i = 1, 2, . . . , n of each of the eigenvalues to perturbations was calculated for both assignments. If c_i = 1, an eigenvalue is conditioned perfectly (i.e., the change in the value of the eigenvalue will be of the same size as the perturbation), and as c_i increases so does the sensitivity of the eigenvalue. These individual sensitivities are displayed in Table 3.

Table 3 shows a high correlation of individual eigenvalue sensitivities. In both cases, the dutch-roll mode is most sensitive to perturbations. The iterative technique has produced a slightly better conditioned set of closed-loop eigenvalues than the direct assignment but exhibits worse handling qualities.

The linear gain matrices required to assign the eigenstructure in Table 2 are given in Table 4. Let $\Delta$ be an $(n \times n)$ real matrix with entries corresponding to perturbations applied to the state matrix $A$. The addition of this disturbance matrix to the closed-loop matrix $A_{c,\text{dot}}$ will cause the eigenvalues to shift. To assess the robustness of the designs, such a $\Delta$ is calculated by rounding $K_{IT}$ to one decimal place and forming $A_{c,\text{dot}} = A + BK_{IT,\text{round}}$. The difference between the closed-loop matrix formed from the iterative assignment and $A_{c,\text{dot}}$ yields $\Delta$. The resultant perturbed eigenvalues calculated on applying this same $\Delta$ to both designs are given in Table 5.

The eigenvalues corresponding to the dutch-roll mode have been affected the most by the disturbance introduced. This was expected because the conditioning is worst in this case. However, both sets of eigenvalues have remained in the left-hand half of the complex plane and, thus, correspond to a stable system.

### Table 2

**Closed-Loop Eigenvector Matrices**

<table>
<thead>
<tr>
<th>$v_1, v_2, v_3$</th>
<th>$v_4$</th>
<th>$v_5$</th>
<th>$v_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
</tbody>
</table>

### Table 3

**Individual Eigenvalue Sensitivities (Closed Loop)**

<table>
<thead>
<tr>
<th>$\lambda_i$</th>
<th>$c_i$ (Direct Assignment)</th>
<th>$c_i$ (Iterative Assignment)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.0 ± 1.0</td>
<td>5.93</td>
<td>5.96</td>
</tr>
<tr>
<td>-1.5 ± 1.5</td>
<td>15.93</td>
<td>15.65</td>
</tr>
<tr>
<td>-0.05</td>
<td>1.12</td>
<td>1.12</td>
</tr>
<tr>
<td>-10</td>
<td>5.06</td>
<td>4.62</td>
</tr>
<tr>
<td>-15</td>
<td>2.79</td>
<td>2.61</td>
</tr>
</tbody>
</table>

### Table 4

**Gain Matrices**

$K_{IT} =$

\[
\begin{bmatrix}
0.0245 & -0.0141 & 0.0946 & -0.0452 & 0.0008 & -0.3543 & 0.0442 \\
0.0048 & -0.0073 & 0.0845 & 0.1766 & 0.0286 & -0.1357 & -0.0722
\end{bmatrix}
\]

$K_{DIR} =$

\[
\begin{bmatrix}
0.0246 & 0.0021 & 0.0789 & -0.0648 & -0.0028 & -0.3490 & 0.1351 \\
0.0029 & -0.0078 & 0.0784 & 0.1814 & 0.0292 & -0.1017 & -0.0829
\end{bmatrix}
\]

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Conclusion
This paper has presented two alternative approaches to the eigenstructure assignment problem, both of which were found to be easily implementable on the Ctrl-C design package because of the availability of powerful matrix manipulation capabilities. An example was chosen for which desired eigenvector structures are well defined so that the relative strengths of each technique could be evaluated. Both methods gave well-conditioned solutions to the problems and, although the iterative design is computationally more expensive, less advance information is necessary.

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References

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